D.E. Liebscher    F/Phys 8 (1978) 131

"Derivation of Einstein's Velocity addition
Theorem thro' use of the invariant
Double Ratio"

refers also to his book
Relativitätstheorie mit Zirkel und
Lineal — 1977

# Transparencies

① Galilean Boosts in Velocity Space.

② Lorentz Boosts in Velocity Space.

③, (3a) cont. of ②

④ The Cayley Metric in Velocity Space

⑤ Chronological Table of the Main papers.

(1a) Geometry in ~~Newton~~ ~~Galilean,~~ classical physics, Special Relativity

| | Galileo | Einstein |
|---|---|---|
| Space-Time | N-E | N-E |
| Space | E | E |
| Velocity Space | E | N-E. |

P. Ehrenfest Gleichförmige Rotation starrer Körper (rigid)
und Relativitätstheorie
Phys. Z. 10 (1909) 918.

— points out 1) $2\pi R' < 2\pi R$. for circ. of
moving circle. but no mention of Non-Euclidean
Geometry. also contradicts 2) $R' = R$. Resolution of
this anomaly is Non-Euclidean geometry on the disc,
but this is not explicitly pointed out by Ehrenfest,
who poses the result as a paradox.

L.E.J. Brouwer 1909. The Nature of
Geometry: Inaugural lecture as Privaat-docent
in the University of Amsterdam 12. Oct. 1909.
p. 114 Collected works remarks: a application
of Lob. Geom to velocity space — Euclidean in
classical mechanics, in relativity Lobachevskian with a
very small curvature — there with vel. of
light becomes infinite in this geometry, so
velocities surpassing the velocity of light cannot be
realized
note added in ~~proof~~: 1919: general relativity should
also be mentioned

Vit. V. Varičak Phys. Z. 11 (1910) 93–96, 287–293

Jahresber. Deutsch Math. Ver. 21 (1912) 102–127

J. Le Roux Journal de Mathématiques
(9) 1 (1922) Fasc. II. 205–253

C. R. Acad. Sci. Paris 203 (1936) 1236–1238.

J. Abelé C. R. 235 (1952) 1007–1008.
P. Malvaux " " " 1009–1011
J. Abelé Revue. gen. Sci. 64 (11-12) (1957),
5. J. R 715 0 0. 365–375

J. Abelé and P. Malvaux
Vitesse et Univers ~~e~~ Relativiste.
Paris (1954)

P. Ehrenfest Phys. Z. 10 (1909) 918 (non-Euclidean geometry due.)

G. Herglotz  Ann der Physik. 31 (393) 1910.

“on ~~the~~ bodies which may be called rigid from the stand point of the Relativity Principle”

written 1. Dec. 1909  submitted 7. Dec. 1909.

Section III discusses The Lorentz transformations and the hyperbolic geometry in $R_3$. — connects with the Cayley metric

the work is referred to in the second paper by Varićak — p.409: “the Lorentz transformation that coincides 1:1 with hyperbolic motions in $R_3$”.

on p 407 the embedding of the Lorentz group by Möbius transformations in the stereographic projection of the light sphere is announced as well.

from Leipzig.

He refers to Ehrenfest, but not to Sommerfeld — He uses his results to discuss the shape of moving rigid bodies.

— also refers to ~~an~~ elegant one of quaternions (in a footnote) to represent the Möbius transformation approach.

F. Klein on the geometrical Foundations of the Lorentz Group Phys. Z. (1911) 17. reprint of [partly?] notes in Jahresbericht der Deutschen Mathematiker-Vereinigung. (1910)

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deals with projective space $x_1 x_2 x_3 x_4 x_5$ – then develops Euclidean geo Riemannian geometry in 4-dimensions in same style as in Thomas's notes. I.P. we specialize affine group (fixed plane) to transformations which leave a hyperquadric in this plane invariant

Further gives interpretation of Lorentz transformations in terms of quaternions.

J. Le Roux 'Relativité restreinte et Géométrie des systèmes ondulatoires' Journal de Mathématiques Pures et appliquées.
Series 9 vol 1 (1922) part II pp 205-253

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Section 27 is entitled Lorentz transformations and geometry of Cayley. — obtains $\theta = \frac{1}{2}\log\frac{1+v}{1-v}$ i.e. is given by cross-ratio $\theta = \frac{1}{2}\log(NN'P_1P_2)$ i.e. is the Cayley distance formula. So "Cayley's geometry furnishes a very simple model of the conservation of 'pseudo-spheres' — refers to projective formulation of Lorentz transformations
Section 29 Le Roux attributes relation 1871 of Cayley's geometry to Klein model of Lobachevsky. — no mention of Varičak
— General tone is critical of Einstein.
— compares Lorentz time with vulgar time.
no reference to Pauli Encycl article.

Le Roux thinks Einstein's results all come from an "absurd" convention about measuring time which is designed to force the invariance of c — It is absurd says Le Roux to extend the theory to all other phenomena without exception.

Le Roux talks (p. 207) of an "extension abusive" Einstein's method of regulating clocks he refers to as "procédé bizarre". (p. 207)

J. Abelé 'Descartes, Precurseur de la Cinématique Relativiste' Revue General des Sciences 64 (1957) 365–75.

Descartes says vel. of light is ∞ rel. to all frames — cf Einstein vel. of light = c in all frames

— refers to his book with Malvaux (1952)

Abelé, Malvaux do not refer explicitly to non-Euclidean geometry.

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Von V. Varićak "Anwendung der Lobatschefskijschen Geometrie in der Relativtheorie" Phys. Z. 11 (1910) 93–96.
written Agram 8th Jan. Received 19th Jan. 1910 published 1st Feb.

'In Relativity exists an absolute velocity' in L. Geom. an absolute length'

— based on earlier paper of Sommerfeld: 'Über die Zusammensetzung der Geschwindigkeit in der Relativtheorie' Phys. Z. 10 (1909) p. 826–829

p. 70

Sommerfeld showed Einstein addition formulas related to spherical Trig. with an imaginary term

$\phi = \phi_1 + \phi_2$ above. $tg\,\phi = i\,v/c$ etc

Submitted 30/Sept/1909, published 10th Nov. 1909.

V. Varičak 'Die Relativtheorie und die Lobatschefskijsche Geometrie'. Phys. Z. 11 (1910) 287–293. (Agram – Yugoslavia). written 12/Feb. submitted to
18/Feb – published 1 April
i.e. Zagreb.

refers the 1903 paper in Yugoslav Academy: connecting Lorentz-FitzGerald Contraction with non Euclidean Geometry!

Substitution $v/c = th\,u$ is due to the Connection.

N.B. Varičak is given a hyperbolic interpretation of velocity addition following Sommerfeld — not really concerned with geometry of different velocities in same frame as is Herglotz etc.

A. Einstein  Zum Ehrenfestschen Paradoxon.
Phys. Z. 12 (1911) 509–510.

Is concerned to demonstrate the ~~rotating~~ reality of the Lorentz contraction — gives example of 2 rulers passing each other in opposite directions as an 'absolute' effect.

Chiu and Hoffmann (eds.)
Gravitation and Relativity 1964

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Good essay by Anderson on Riemannian geometry and symmetry, conservation laws in G.R. and on quantization of G.R.

p.180 2 parts to Noether's theorem
a) p-parameter Lie groups and ~~[illegible]~~ gauge groups [illegible] of arbitrary functions
In second case conservation laws become identities — refers to use of [illegible] identities
— draws on Bergmann P.R. 112, (1958), 287
does relation between the 2 sets.

Wheeler discusses measurement of intervals in space-time by Marzke's "[illegible] method"

Also chapter on Geometrodynamics, "Mechanics -- (by) -- 2 kinds, kinematics and dynamics. Kinematics describes conceivable motions without asking whether they are allowed or forbidden. Dynamics makes the difference between a physically realizable and a conceivable history.

Recall good folio 2, Dirac on the Many Bodies of Rod – scalar field can. deriv. of Schwarzschild soln so as to present boundary conditions at ∞ from 'whirling'

Stephenson & Kilmister 'Special Relativity for Physicists', 1958 – ch I gives ref. alternatives to classical Lorentz transformations Whitehead, Pars, Eddington, Romain & Iwatsuki Whitrow, Tonnelat, Reid, Skinner, Ruse, Milne Frank, Rothe and Singleton.

– notes good pt that invariance of wave eq ⇏ inv. Maxwell's Eqs – must be proved as an extra

p.33 Transformation of rels. reach 4 vel. $\gamma, \underset{\wedge}{u}\gamma$

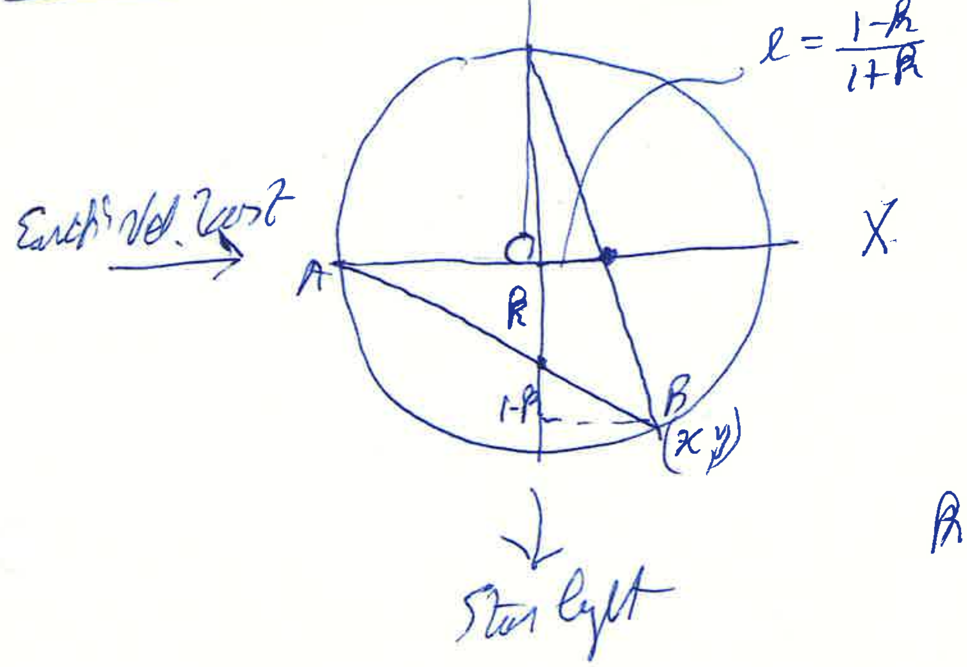
$$\gamma' u' = \gamma(\beta u^x - v\gamma)$$

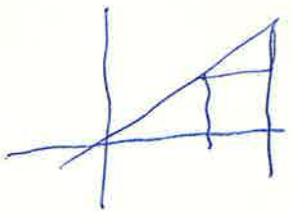
$$\gamma' = \gamma(\gamma - \gamma . v.u/c^2)$$

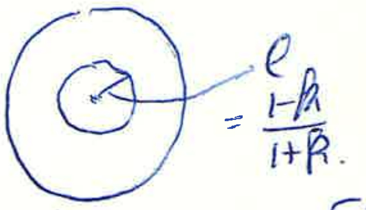
$$\therefore \beta' = \beta^2(1 - u.v/c^2)$$

and $u' = \dfrac{u^x - v}{1 - u.v/c^2}$ as required

# Aberration circle


Y
X
$l = \frac{1-R}{1+R}$
Earth's Vel. West
A
O
R
1-R
B
(x, y)
Star light





$R = \frac{\sqrt{1-\beta^2}}{1+\beta}$ , $\therefore l = \underline{\underline{\frac{1+\beta-\sqrt{1-\beta^2}}{1+\beta+\sqrt{1-\beta^2}}}}$

$l = \frac{x}{1-y}$

we have $\frac{l}{1} = \frac{x}{1-y}$

Eq of AB is $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$

with $(x_1, y_1) = (-1, 0)$ $(x_2, y_2) = (0, -R)$

$\therefore \frac{y}{x+1} = \frac{-R}{}$ $\therefore y = -R(1+x)$

Intersects $x^2+y^2=1$ at $R^2(1+x)^2 + x^2 = 1$

or $R^2 + (R^2+1)x^2 + 2R^2x - 1 = 0$

or $(1+R^2)x^2 + 2R^2x - (1-R^2) = 0$.

$\therefore (x+1)\left((1+R^2)x - (1-R^2)\right) = 0$

$\therefore x = -1$ or $x = \frac{1-R^2}{1+R^2}$ $y = -R\left(1 + \frac{1-R^2}{1+R^2}\right) = \frac{-2R}{1+R^2}$

$\therefore l = \frac{1-R^2}{1+R^2} \Big/ 1 + \frac{2R}{1+R^2} = \frac{1-R^2}{(1+R)^2} = \underline{\underline{\left(\frac{1-R}{1+R}\right)}}$

## Conformal transformation of $S^2$ for rel. light rays ①

$$z = \frac{x_1 + i x_2}{1 - x_3}$$

$$x_1 = \frac{z + \bar{z}}{1 + |z|^2}$$

$$z' = \frac{x_1' + i x_2'}{1 - x_3'}$$

$$x_2 = \frac{z - \bar{z}}{i(1 + |z|^2)}$$

$$x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$$

also for rel. boost. z - axis. 3 ∥ vel.

$$x_3' = \frac{x_3 - v}{1 - v . x_3}$$

$$x_2' = \frac{x_2 \sqrt{1 - v^2}}{1 - v x_3}$$

$$x_1' = \frac{x_1 \sqrt{1 - v^2}}{1 - v x_3}$$

$$\therefore \quad x_3' = \frac{\dfrac{|z|^2 - 1}{|z|^2 + 1} - v}{1 - \dfrac{v(|z|^2 - 1)}{|z|^2 + 1}}$$

$$x_1' = \frac{(z + \bar{z}) \sqrt{1 - v^2}}{(1 + |z|^2)\left(1 - v . \dfrac{|z|^2 - 1}{|z|^2 + 1}\right)}$$

$$x_2' = \frac{(z - \bar{z}) \sqrt{1 - v^2}}{\left(1 - \dfrac{v(|z|^2 - 1)}{|z|^2 + 1}\right) i\,(1 + |z|^2}$$

$$\therefore\ x_1' = \frac{(z+\bar{z})\sqrt{1-v^2}}{1+|z|^2 - v(|z|^2-1)} = \frac{(z+\bar{z})\sqrt{1-v^2}}{(1+v)+|z|^2(1-v)} \quad ②$$

$$x_2' = -i\,\frac{(z-\bar{z})\sqrt{1-v^2}}{\cancel{s}\,(1+v)+|z|^2(1-v)}$$

$$x_3' = \frac{|z|^2(1-v)-(1+v)}{(1+v)+|z|^2(1-v)}$$

$$\therefore\ \overset{\cancel{x}}{z}{}' = \frac{\dfrac{\sqrt{1-v^2}}{(1+v)+|z|^2(1-v)}\left[z+\bar{z}+z-\bar{z}\right]}{1-\dfrac{|z|^2(1-v)-(1+v)}{(1+v)+|z|^2(1-v)}}$$

$$= \sqrt{1-v^2}\,\frac{2z}{(1+v)+|z|^2(1-v)-|z|^2(1-v)+(1+v)}$$

$$= \frac{2\sqrt{1-v^2}\cdot z}{2(1+v)} = \frac{\sqrt{1-v^2}\,z}{\underline{\underline{(1+v)}}} = \sqrt{\frac{1-v}{1+v}}\cdot z$$

now take vel $\parallel$ x-axis ③

$$\text{So } x_1' = \frac{x_1 - v}{1 - v x_1} \qquad x_2' = \frac{\sqrt{1-u^2}\, x_2}{1 - u x_1}$$

$$x_3' = \frac{\sqrt{1-u^2}\,.\, x_3}{1 - u x_1}$$

$$\text{So } x_1' = \frac{\dfrac{(z+\bar{z})}{1+|z|^2} - v}{1 - \dfrac{v(z+\bar{z})}{1+|z|^2}} = \frac{z+\bar{z} - v(1+|z|^2)}{1+|z|^2 - v(z+\bar{z})}$$

$$x_2' = \frac{\sqrt{1-v^2}\,.\,\dfrac{z-\bar{z}}{i(1+|z|^2)}}{1 - v.\dfrac{z+\bar{z}}{1+|z|^2}} = \frac{\sqrt{1-u^2}\,(z-\bar{z})}{i\left[1+|z|^2 - u(z+\bar{z})\right]}$$

$$x_3' = \frac{\sqrt{1-u^2}\,.\,\dfrac{|z|^2-1}{|z|^2+1}}{1 - v.\dfrac{z+\bar{z}}{1+|z|^2}} = \frac{\sqrt{1-u^2}\,(|z|^2-1)}{1+|z|^2 - u(z+\bar{z})}$$

$$\therefore z' = \frac{z+\bar{z} - u(1+|z|^2) + \sqrt{1-u^2}\,(z-\bar{z})}{\cancel{\sqrt{1-u^2}\,(|z|^2-1)}\; 1+|z|^2 - u(z+\bar{z}) - \sqrt{1-u^2}\,(|z|^2-1)}$$

④

$$\therefore z' = \frac{z+\bar{z} - u(1+|z|^2) + \sqrt{1-u^2}\,(z-\bar{z})}{1+|z|^2 - u(z+\bar{z}) - \sqrt{1-u^2}\,(|z|^2-1)}$$

$$= \frac{z(1+\sqrt{1-u^2}) + \bar{z}(1-\sqrt{1-u^2}) - u z\bar{z} - v}{(1+\sqrt{1-u^2}) + z\bar{z}(1-\sqrt{1-u^2}) - uz - u\bar{z}}$$

$$= N/D$$

$$= \cancel{\zeta} z \quad \cancel{(z+a)(\bar{z}+u)}$$

$$N = \zeta \cdot z\left(1+\sqrt{1-u^2} - v\bar{z}\right) + \left(\frac{1-\sqrt{1-u^2}}{v}\right) \cdot v\bar{z} - (1+\sqrt{1-u^2})\frac{v}{1+\sqrt{1-u^2}}$$

---

$$ax + by + cxy + d.$$

$$= x(a+cy) + (d+by)$$

$$\subset b(d+cy)$$

$$= (x+b)(a+cy)$$

$$\frac{u^2}{1-\sqrt{1-u^2}} = 1+\sqrt{1-u^2}$$

$$u^2 = 1-(1-u^2)$$

---

$$D = z\left((1-\sqrt{1-u^2})\bar{z} - u\right) + (1+\sqrt{1-u^2}) - u\bar{z}$$

$$= z\left((1-\sqrt{1-u^2})\bar{z} - u\right) + \frac{v}{1-\sqrt{1-u^2}}\left\{(1-\sqrt{1-u^2})\bar{z} - v\right\}$$

$$= \left[(1-\sqrt{1-u^2})\bar{z} - v\right] \times \left[z - \frac{v}{1-\sqrt{1-u^2}}\right]$$

$$N = \bar{z}\left(1-\sqrt{1-v^2}\right) - v$$

$$+ z\left(1+\sqrt{1-v^2} - v\bar{z}\right)$$

$$= \bar{z}\left(1-\sqrt{1-v^2}\right) - v$$

$$+ \frac{1+\sqrt{1-v^2}}{v} . z \left[ v - \bar{z}\left\{\frac{v^2}{1+\sqrt{1-v^2}}\right\}\right]$$

149.53
120.18
192.50

$$= \bar{z}\left(1-\sqrt{1-v^2}\right) - v$$

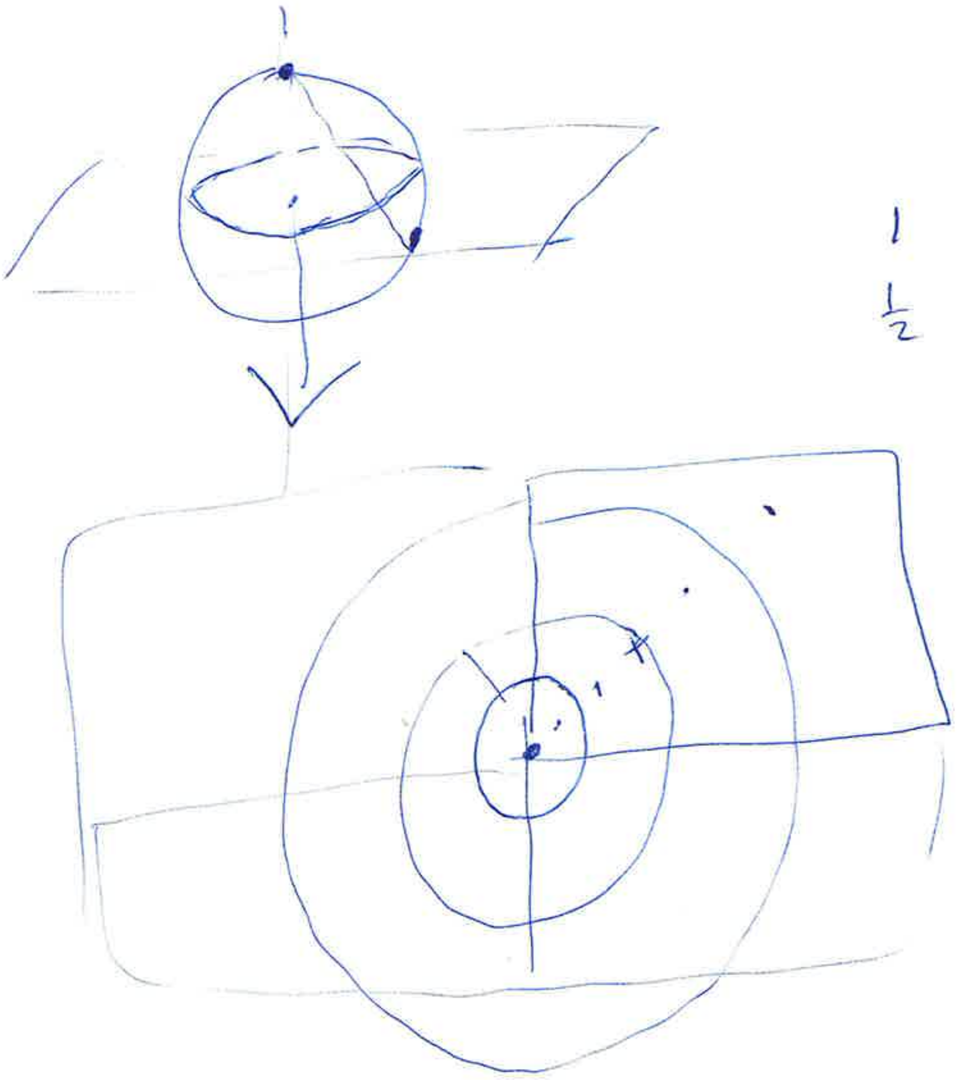
$$- \frac{1+\sqrt{1-v^2}}{v} z \left\{\frac{v^2\bar{z}}{1+\sqrt{1-v^2}} - v\right\}$$

$$= \left(\bar{z}\left(1-\sqrt{1-v^2}\right) - v\right) - \frac{1+\sqrt{1-v^2}}{v} z \left(1-\sqrt{1-v^2} - v\right\}$$

$$= \left(\bar{z}\left(1-\sqrt{1-v^2}\right) - v\right)\left(1 - \frac{1+\sqrt{1-v^2} . z}{v}\right)$$

$$\therefore z' = \frac{N}{n} = \frac{1 - \frac{1+\sqrt{1-v^2}}{v} z}{z - \frac{v}{1-\sqrt{1-v^2}}} = \frac{A-Bz}{z-C}$$

So $z' = kz = \frac{\sqrt{1-\beta^2}}{1+\beta} \cdot z$



| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| $\frac{1}{2}$ | 1 | 3/2 | |

Searchlight effect

Lorentz Tran. from Vel. addition

$$\xi' = \frac{X'}{T'} = \frac{\frac{X}{T} - v}{1 - \frac{X}{T}\frac{v}{c^2}} \qquad \xi' = \frac{\xi - v}{1 - v/c^2\,\xi}$$

$$\eta' = \frac{\eta\,.\sqrt{1-v^2/c^2}}{1 - v/c^2\,\xi}$$

$$\eta' = \frac{Y'}{T'} = \frac{\frac{Y}{T}\sqrt{1-v^2/c^2}}{1 - \frac{X}{T}v/c^2}.$$

$$\xi' = \frac{A\xi + B}{C\xi + D}$$

$$\eta' = \frac{F\eta}{A + B\xi}$$

∴ If we assume. $Y' = Y.$

$$\frac{1}{T'} = \frac{\sqrt{1-v^2/c^2}/T}{1 - x/t\;v/c^2}.$$

$$T' = ~~kk(~~\;(T - x\,v/c^2)\,k.$$

$$x' = \underline{x - vt} \times \frac{T'}{T - xv/c^2} = k(x - vt)$$

Projective transformations in rel. Speed ①

$$U_x' = \frac{aU_x + b}{eU_x + d} = \frac{U_x + b'}{e'U_x + d'}$$

$$U_y' = \frac{U_y}{eU_x + d} = \frac{aU_y}{e'U_x + d'}$$

Put $U_x = 0$ then $U_x' = v$

$\Rightarrow$ ~~$U_x'$~~ $v = \frac{b'}{d'}$ or

Solve for $U_x$ in terms of $U_x'$

Put $U_x' = 0$ then $U_x = -v$

$$0 = \frac{-v + b'}{d' - ev} \Rightarrow b' = v \quad \text{and } d'^2)$$

$$\therefore U_x' = \frac{U_x + v}{1 - KU_x)^2} \qquad U_x = c \Rightarrow K = c/c^2.$$

$$U_y' = \frac{aU_y}{1 - KU_x'}$$

$$U_x'^2 + U_y'^2 = \frac{(U_x - v)^2}{(1- \quad)^2} + \frac{a^2U_y^2}{(1- \quad)^2}. \quad = c^2.$$

if $U_x^2 + U_y^2 = c^2$ -

$$u_x' = \frac{u_x - v}{1 - \frac{v u_x}{c^2}}$$

$$u_y' = \frac{u_y \sqrt{1 - v^2/c^2}}{1 - \frac{v u_x}{c^2}}$$

$$u' = \frac{k' u_y}{1 - k'' u_x}.$$

$$u_x' = \frac{A u_x + + B u_y + C}{D u_x + E u_y + F}.$$

$$u_x' = \frac{u_x - v}{1 - k(v) u_x}$$

$$c = \frac{c - v}{1 - kc}$$

$u_x$ $u_x = c$

$u_x' = c$

So $c - kc^2 = c - v$.

$kc^2 = v$

$k = c/v^2$.

$$u_x' =$$

---

$$u_x' = \frac{\alpha u_x + \beta}{\gamma x + \delta}$$

$$u_y' = \frac{\varepsilon u_y'}{\gamma' u_{\cancel{x}}' + \delta}$$

$$x' = ax + vt$$

$$y = y'$$

$$t' = a' x + v' t.$$

$$\frac{x'}{t'} = \frac{a\left(\frac{x}{t}\right) + v}{a'(x/t) + v'}$$

$$\frac{y'}{t'} = \frac{y/t}{\alpha\left(\frac{x}{t}\right) + \beta'}$$

Transformation of the light circle

$$u_x' = \frac{u_x - v}{1 - v u_x/c^2}$$

$u_x = c\cos\theta$

$u_y = c\sin\theta$

$$u_x' = \frac{c\cos\theta - v}{1 - \cancel{c}\cos\theta\, v/c}$$

$$= c\left(\frac{c\cos\theta - v}{c - v\cos\theta}\right)$$

$\frac{\theta = 0}{\theta = \pi}$ $u_x' = +c$, $u_x' = -c$

Put $v = c$

$$u_x' = c\left(\frac{c\cos\theta - c}{c - c\cos\theta}\right) = -c$$

$$u_y' = \frac{u_y\sqrt{1 - v^2/c^2}}{1 - v u_x/c^2}$$

$$u_y' = \frac{\overset{c\sin\theta}{\cancel{\theta}}\sqrt{1 - v^2/c^2}}{1 - \cos\theta \cdot v/c}$$

$$= \frac{c^2\sin\theta\sqrt{1 - v^2/c^2}}{c - v\cos\theta}$$

$u_x' = 0$

$u_y' = c$

$u_y' = 0$ except at $\theta = 0$

also $u_y' = \frac{\sin\theta}{1 - \cos} \sim \frac{1}{\theta^2}$

$\sin\theta\sqrt{\ }$ $v = c - \varepsilon$

$$\frac{\sin\theta\sqrt{1 - \frac{(c-\varepsilon)^2}{c^2}}}{c(1 - \cos) + \varepsilon \cos}$$

Put $\theta = 0$

$u_x' = c\left(\frac{c-v}{c-v}\right)$

then $u_x' = +c$, $u_y' = 0$

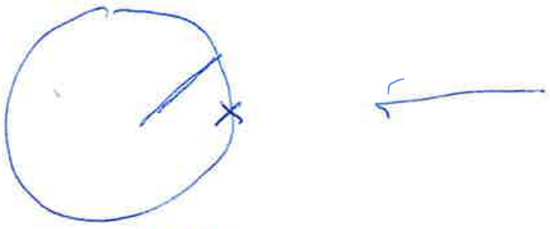
$\theta = \pi$

$u_x' = c\left(\frac{-c-v}{c+v}\right) = -c$

But for $\theta \neq 0$ and $v = c$

$$u_x' = c\left(\frac{c\cos\theta - c}{c - c\cos\theta}\right) = -c$$

$$u_y' = 0$$



write $v = c-\beta$

For $\theta$ small $= \varepsilon$ say.

$$u_x' = c\left(\frac{c\cos\varepsilon - v}{c - v\cos\varepsilon}\right)$$

$$= c\left(\frac{c(1-\frac{1}{2}\varepsilon^2) - v}{c - v(1-1/2\,\varepsilon^2)}\right) = c\left[\frac{c - v - \frac{1}{2}c\varepsilon^2}{c - v + \frac{1}{2}v\varepsilon^2}\right]$$

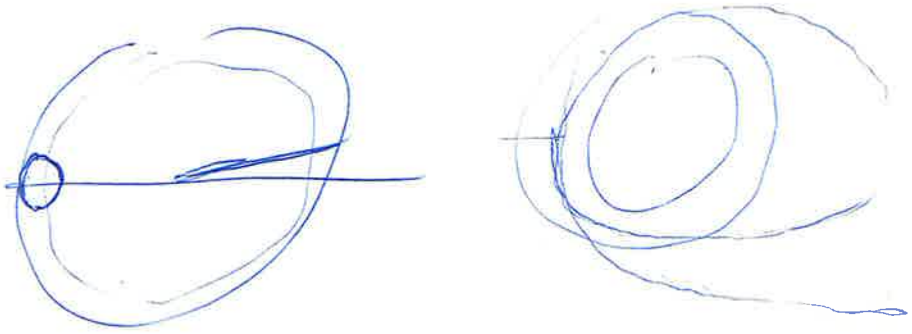
$$\rightarrow c\left[\frac{\beta - \frac{1}{2}c\varepsilon^2}{\beta + \frac{1}{2}(c-\beta)\varepsilon^2}\right] = c\left[\frac{\beta - \frac{1}{2}c\varepsilon^2}{\beta + \frac{1}{2}c\varepsilon^2 - \beta\varepsilon^2}\right]$$

$$= c\left[\frac{\beta - \frac{1}{2}c\varepsilon^2}{\beta(1-\varepsilon^2) + \frac{1}{2}c\varepsilon^2}\right]$$

For fixed $\varepsilon$

we' [illegible] negative then $\beta < \frac{1}{2}c\varepsilon^2$

But for $\varepsilon = 0$ we can never get $\beta < \frac{1}{2}c\varepsilon^2$



Then for fixed $\beta$

we can ~~[illegible]~~ always choose an $\varepsilon \geq 0$

that $\beta - \frac{1}{2}c\varepsilon^2$ remains positive.

$y^2 = 4ax.$

$y^2 = 4ax + c$

$ax^2 + by^2 + 2hxy + \ldots$

$ab - h^2 = 0$

$ab = 0.$

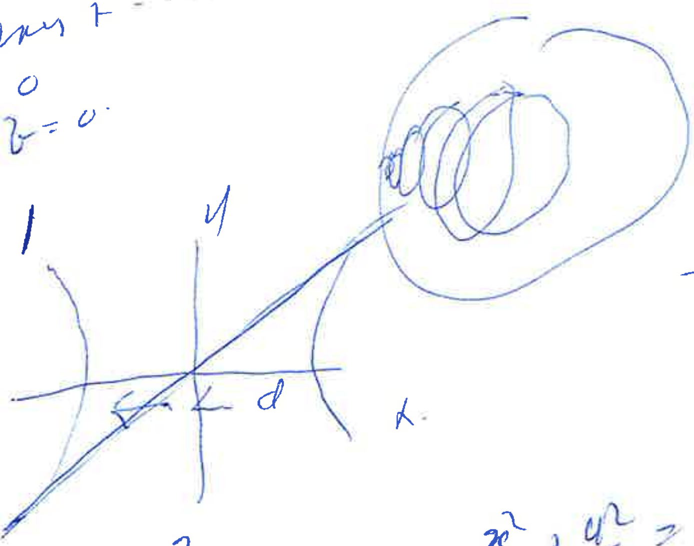
$\frac{(x-l)^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{x^2}{a^2} - \frac{2lx}{a^2}$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$y = b \quad \frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$

$\frac{b}{a^2}$

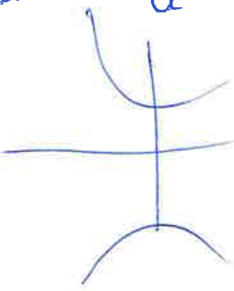


$\frac{x}{a} = \frac{y}{b}$

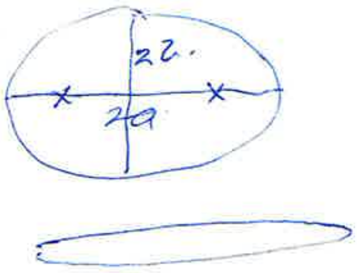
$y = \frac{b}{a}x$

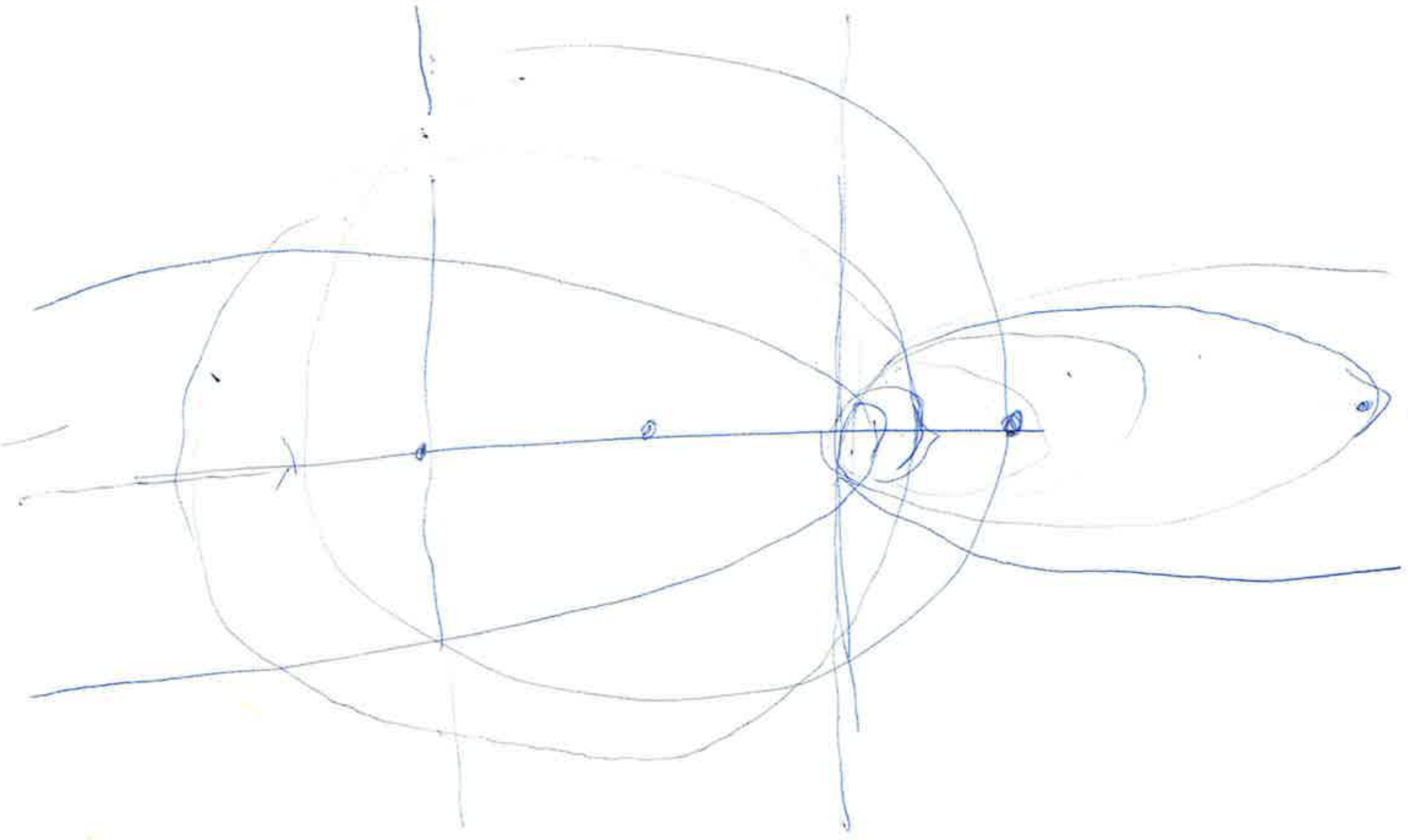
slope. $= \frac{b}{a}$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$





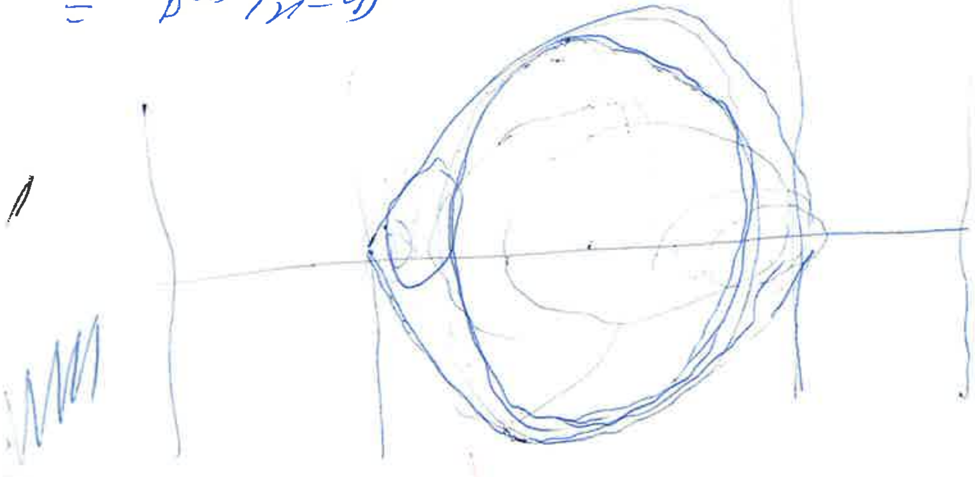
$v = \frac{c^2}{u_x}$

$\{\phi(x), \phi(y)\} = D^{(1)}(x-y)$

$\phi'(x) = \phi(L^{-1}x)$.

$\{\phi'(x), \phi'(y)\} = D^{(1)}(L^{-1}(x-y))$

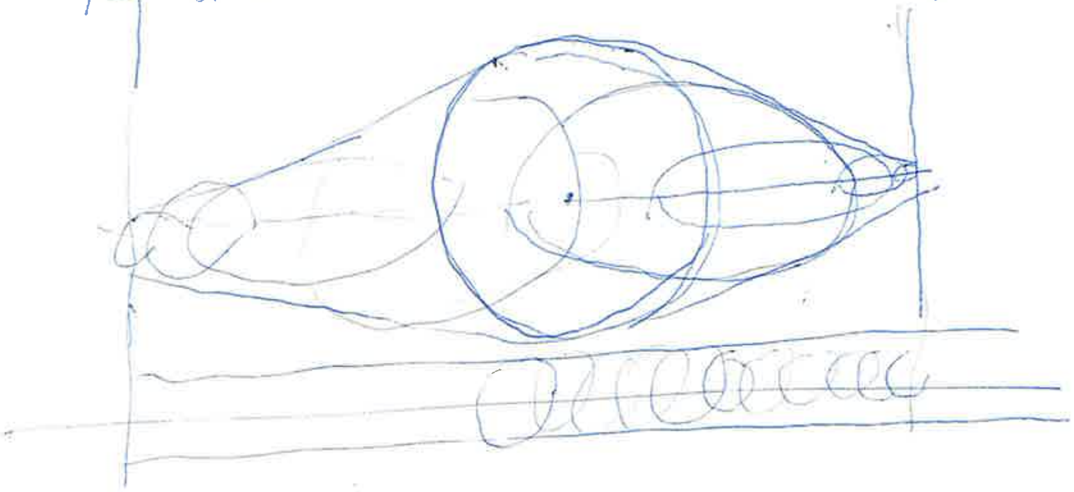
$= D^{(1)}(x-y)$

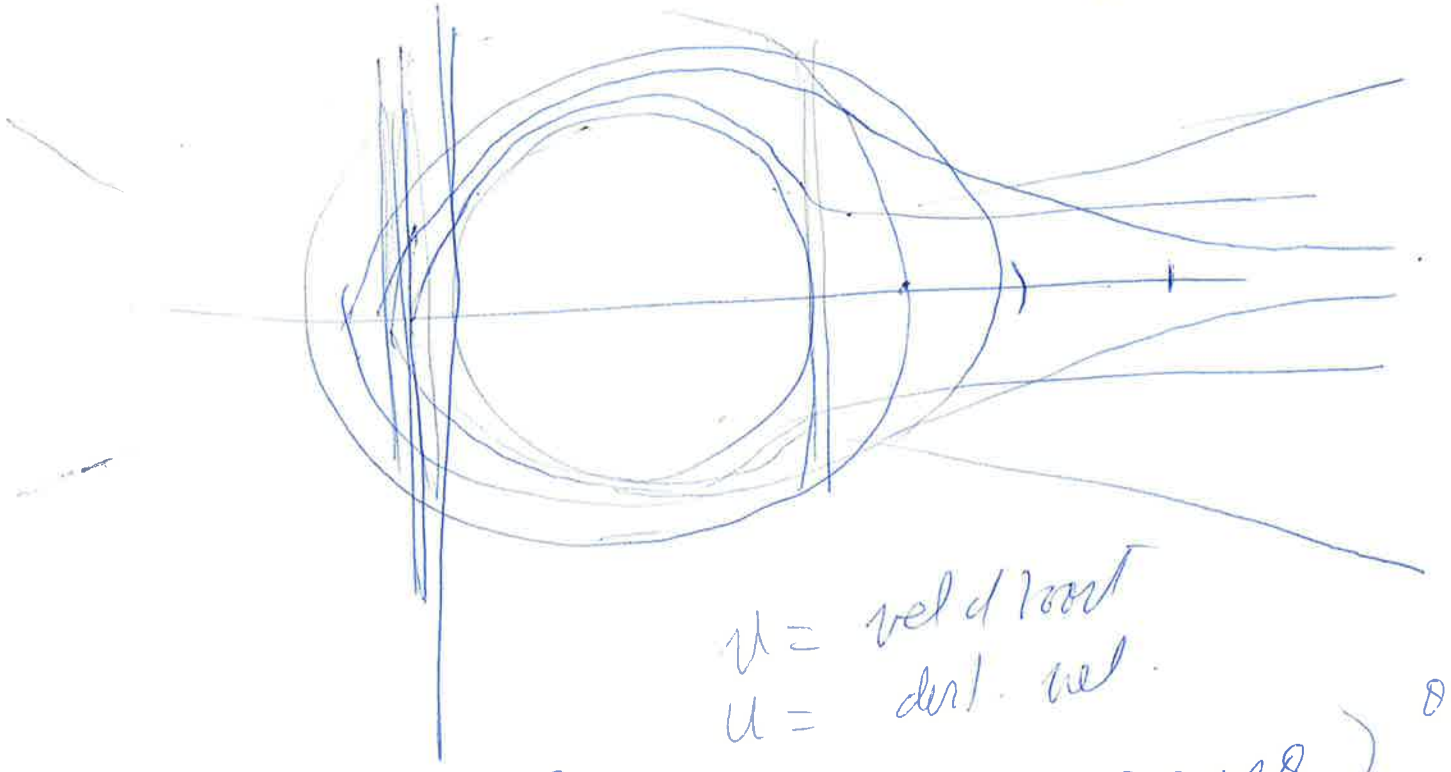


$$u = c \frac{\left(1 - 2v\cos\theta/c + v^2/c^2 - \frac{v^2}{c^2}\sin^2\theta\right)^{1/2}}{1 - \frac{v\cos\theta}{c}}$$

$1 - 2v\cos\theta/c + v^2/c^2\cos^2\theta$

$u'$ $\theta'$





$v =$ vel of boat
$u =$ drl. vel

take $u = 2c$
and $v = \frac{1}{2}c$

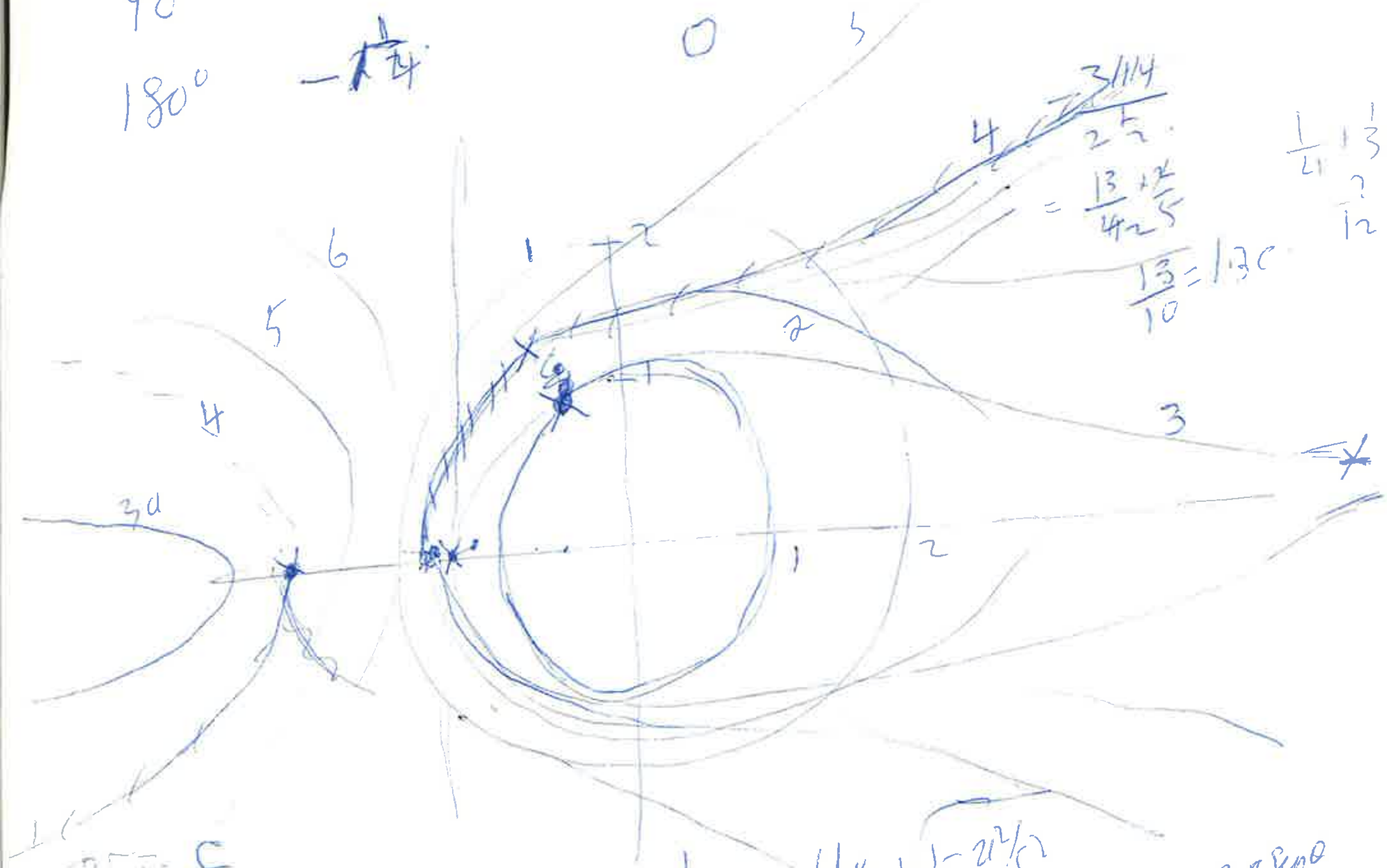
plot $u_x = 2c\cos\theta$
$u_y = 2c\,\mathrm{sen}\,\theta$ $\Big\}$ $\theta = 0, 45, 90^\circ, 135^\circ, 180^\circ$

$$u_x' = \frac{2c\cos\theta - \frac{1}{2}c}{1 - \frac{1}{2}c \cdot \frac{2c\cos\theta}{c^2}} = c\,\frac{\left(2\cos\theta - \frac{1}{2}\right)}{1-\cos\theta}$$

$$u_y' = \frac{c\,\mathrm{sen}\,\theta}{1-\cos\theta}\cdot\sqrt{1-1/4} = \frac{c\,\mathrm{sen}\,\theta\,\sqrt{3}}{2(1-\cos\theta)}$$

| $\theta$ | $u_x'$ | $u_y'$ |
|---|---|---|
| 0 | $\infty$ | 0 |
| 90° | $-\frac{1}{2}$ | $\sqrt{3}/2$ |
| 180° | $-\frac{1}{4}$ | 0 |

$\frac{1}{4} + 3/4$


5
4
3/1/4
2 1/2
= 13/4 2/5
13/10 = 1.30
1/4 + 1/3
7/12
1
2
6
5
4
3a
2
3
1
2
1

$v = c$

$$u_x' = \frac{u_x - c}{1 - u_x/c}$$

$= -c$

$$u_y' = \frac{u_y\sqrt{1 - v^2/c^2}}{1 - u_x/c}$$

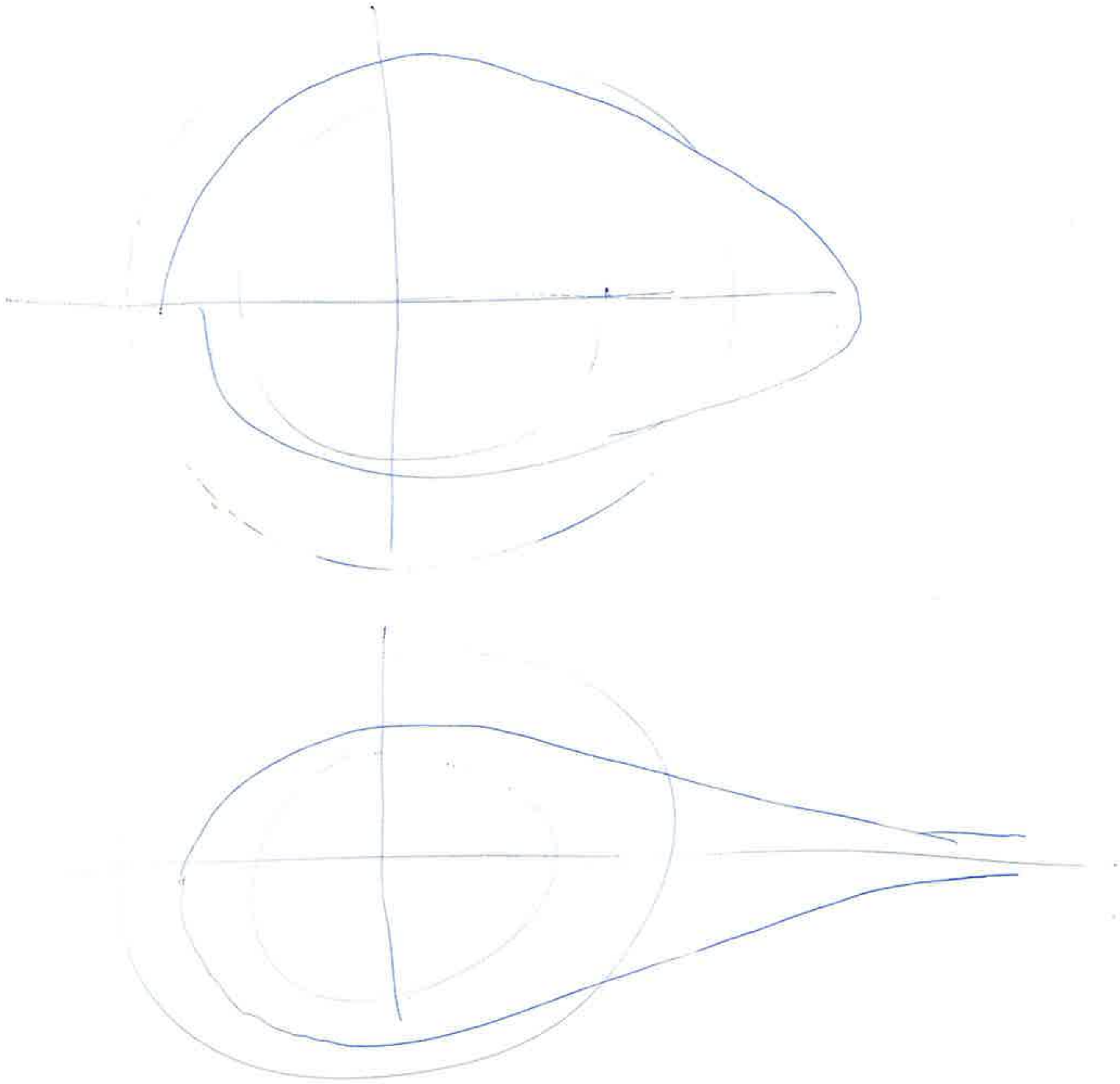
$$u_y' = \frac{\sqrt{6}/4}{1 - 3/2\cos\theta}$$

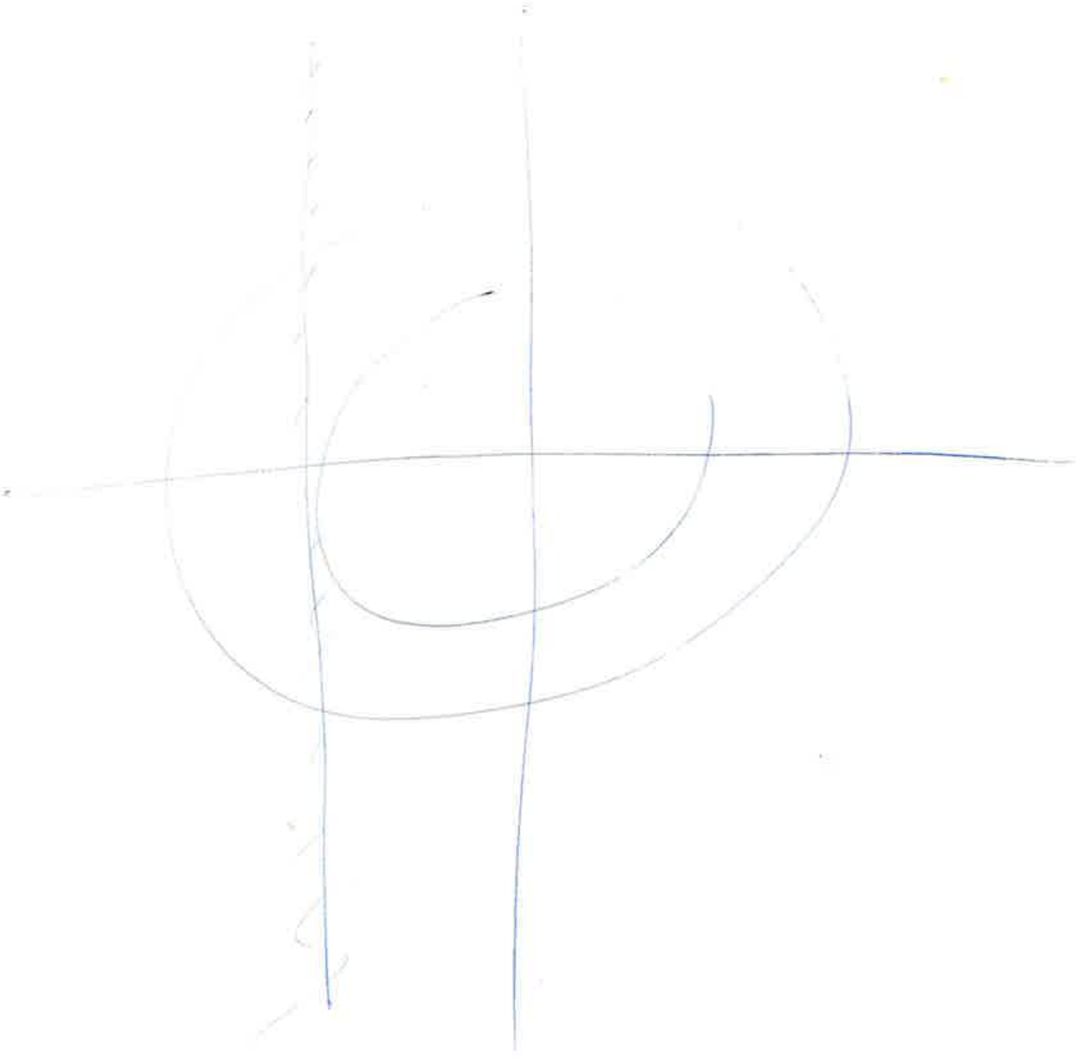
$2c\sin\theta$

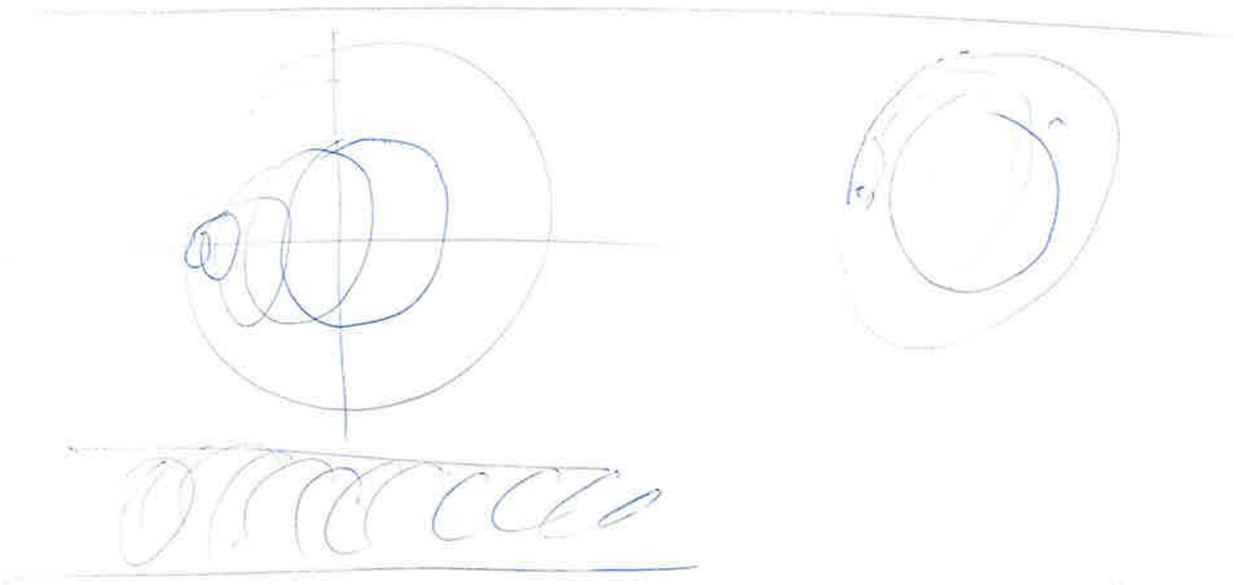
---

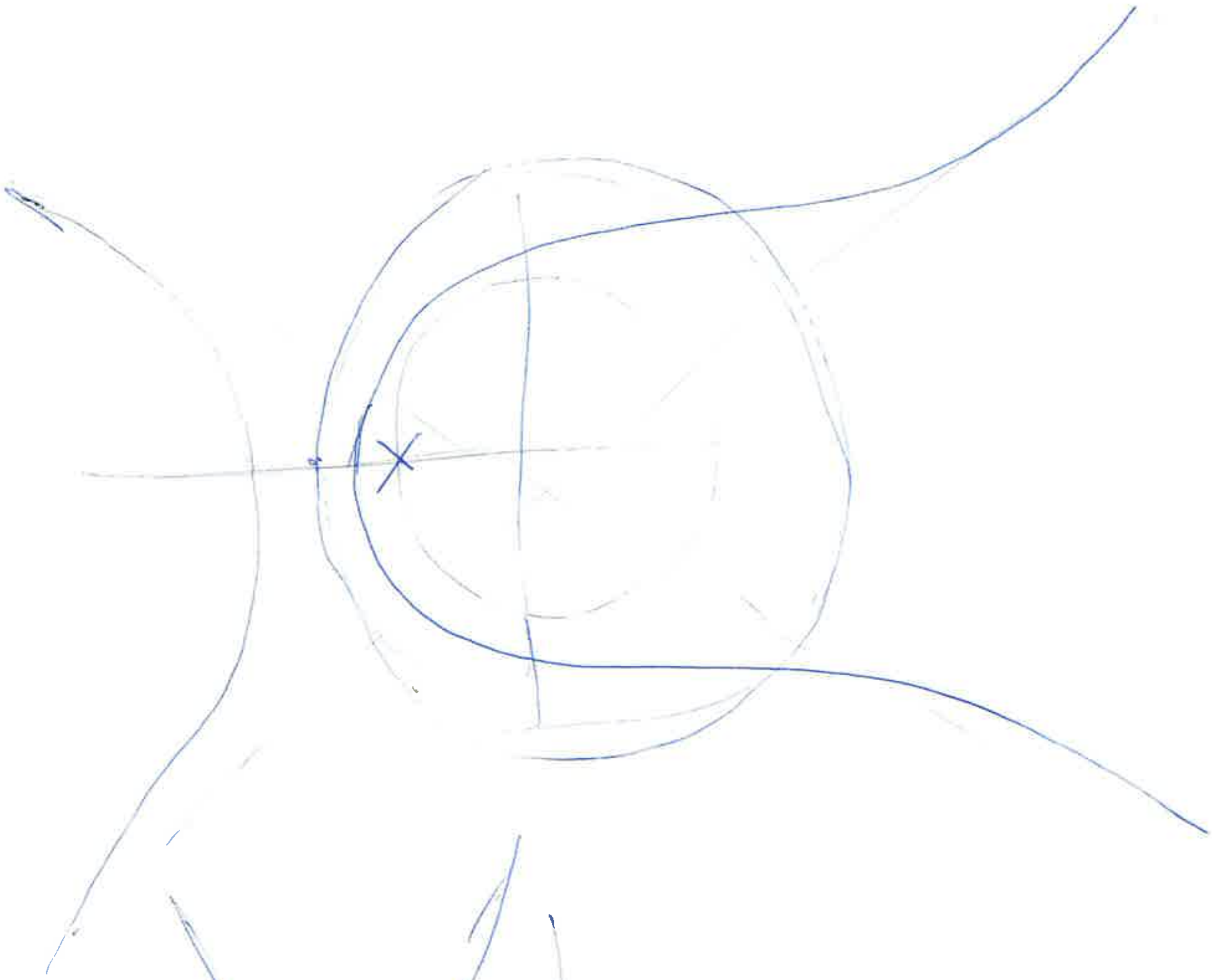
$v = 3/4\,c$ $\quad u_x' = \dfrac{2\cos\theta - 3/4}{1 - 3/4 \cdot 2\cos\theta} = \dfrac{2\cos\theta - 3/4}{1 - 3/2\cos\theta}$

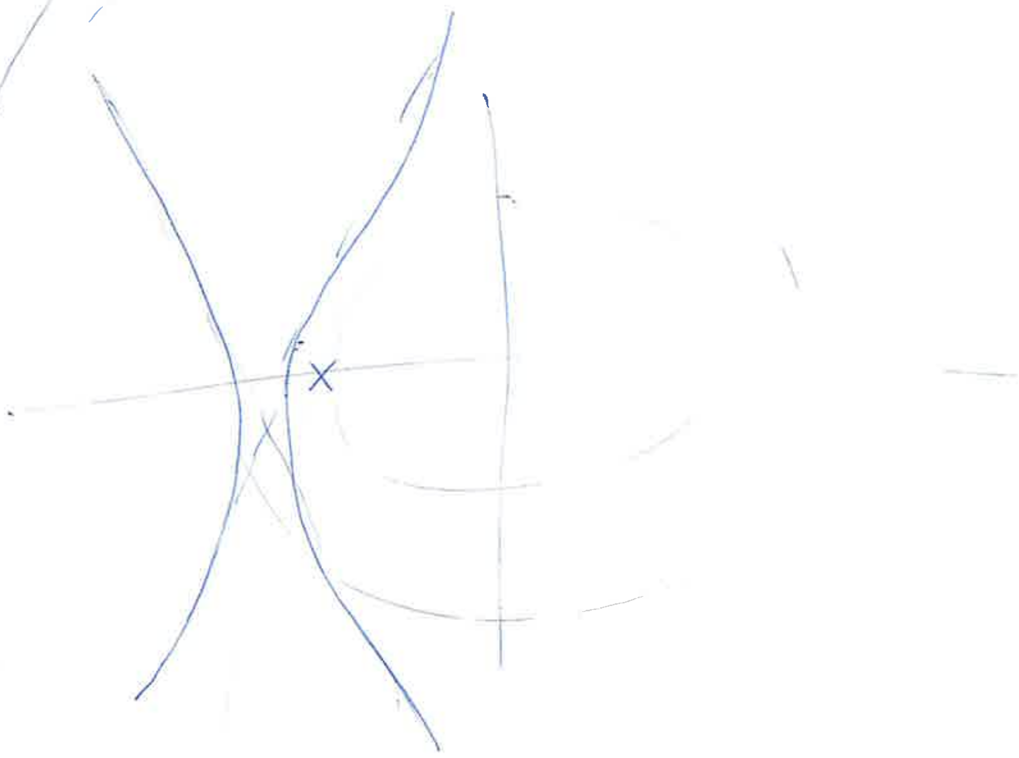
| $\theta$ | $u_x'$ | $u_y'$ |
|---|---|---|
| 0 | $1\frac{1}{4}/-\frac{1}{2} = -2\frac{1}{2}$ | 0 |
| 90° | $-3/4$ | $\sqrt{6}/2$ |
| 180° | $= \frac{2 \cdot 3/4 - 3/4}{1 + 3/2}$ | 0 |
| $\cos^{-1} 2/3$ | $\frac{7/12}{-0}$ | $\frac{+\infty}{-0}$ |











PROOF VEL. Surface is ellipse

$$u_x = \frac{u_x' + v}{1 + u_x' v/c^2} \qquad u_y = \frac{(1 - v^2/c^2)^{1/2} u_y'}{1 + u_x' v/c^2} \quad ①$$

$$x = u_x t \qquad u_x' = w' \cos\alpha \qquad \cancel{w't \quad c/r}$$
$$y = u_y t \qquad u_y' = w' \sin\alpha$$

then $x = \dfrac{(w'\cos\alpha + v)t}{1 + w'\cos\alpha . v/c^2}$ $\qquad y = \dfrac{(1 - v^2/c^2)^{1/2}\, w(\sin\alpha)t}{1 + w'\cos\alpha\, v/c^2}$

$$x + w'\cos\alpha . v/c^2 . x = t.w'\cos\alpha + tv$$

$$\therefore \cos\alpha\left[w' v/c^2 . x - w't\right] = tv - x.$$

$$\therefore \cos\alpha = \frac{x - vt}{w'(t - \cancel{w'}\frac{vx}{c^2})}$$

$$\sin\alpha = \frac{y}{\sqrt{1 - v^2/c^2}\, . w't} \cdot \left[1 + w' v/c^2 \cdot \frac{(x - vt)}{w't - w' vx/c^2}\right]$$

$$= \frac{y}{\sqrt{1 - v^2/c^2}\, . w't} . \left[\frac{w't - \cancel{\frac{w'vx}{c^2}} + \cancel{\frac{w'vx}{c^2}} - \frac{w'v^2t}{c^2}}{w't - w' vx/c^2}\right]$$

$$= \frac{y}{\sqrt{1 - v^2/c^2}\, . \cancel{t}\, w'}\left[\cancel{w'}\, t(1 - v^2/c^2)^{1/2}\right] \Big/ \left(\cancel{w'}\, t - \cancel{w'}\, vx/c^2\right)$$

$$\cos^2\alpha + \sin^2\alpha = 1 = \frac{(x - vt)^2 + \dfrac{y^2\,\cancel{t}}{\cancel{(1 - v^2/c^2)}\,\cancel{t}}\,(1 - v^2/c^2)}{w'^2\,(t - vx/c^2)^2}$$

$$\therefore\ w'^2\left(t^2 + \frac{v^2x^2}{c^4} - \frac{2xvt}{c^2}\right)$$

$$= x^2 + v^2t^2 - 2xvt + y^2(1 - v^2/c^2)$$

$$\therefore\ t^2\left(w'^2 - v^2\right) + x^2\left(\frac{w'^2v^2}{c^4} - 1\right)$$

$$+ 2xvt\left(1 \mp w'^2/c^2\right) - y^2(1 - v^2/c^2) = 0$$

or $$x^2\left(1 - \frac{w'^2v^2}{c^4}\right) + t^2\left(v^2 - w'^2\right)$$ $*$

$$\mp 2xvt\left(1 \mp w'^2/c^2\right) + y^2(1 - v^2/c^2) = 0$$

cf. eqn $x^2 - 2axt + a^2t^2 + by^2 - u^2w'^2t^2 = 0$.

i.e. $x^2 - 2axt + by^2 + t^2(a^2 - u^2w'^2) = 0$

$$* \Rightarrow\ x^2 \mp 2xt\cdot\frac{v\,(1 \mp w'^2/c^2)}{1 - \frac{w'^2v^2}{c^4}} + y^2\cdot\frac{1 - v^2/c^2}{1 - w'^2v^2/c^4}$$

$$+ t^2\left(\frac{v^2 - w'^2}{1 - w'^2v^2/c^4}\right)$$

cf. $$a^2 - u^2w'^2 = \frac{v^2(1 - w'^2/c^2)^2}{(1 - v^2w'^2/c^4)^2} - \frac{(1 - v^2/c^2)^2\cdot w'^2}{(1 - v^2w'^2/c^4)^2}$$

Num$^r$ $\rightarrow$ $(v^2 - w'^2)(1 - v^2w'^2/c^4) = v^2 - \frac{v^4}{c^4}\cdot w'^2 - w'^2 + \frac{v^2w'^4}{c^4}$

③

$$N = u^2\left(1 + w'^4/c^4 - 2w'^2/c^2\right) - w'^2\left(1 + u^4/c^4 - 2u^2/c^2\right)$$

$$= v^2 - w'^2 + u^2\frac{w'^4}{c^4} - w'^2\frac{u^4}{c^4}$$

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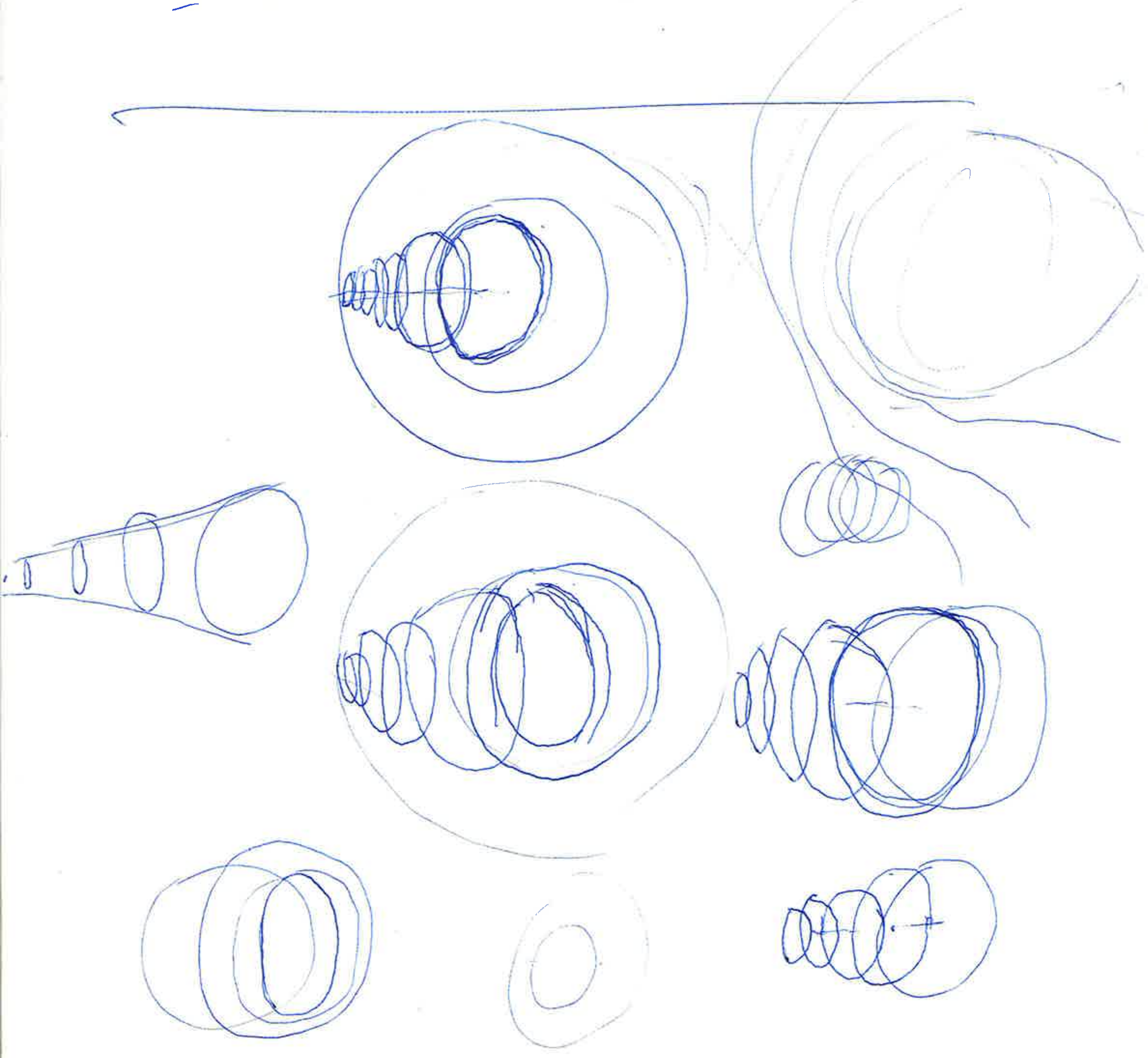

So $\ell = \frac{1-k}{1+k} = \frac{1 - \frac{\sqrt{1-\beta^2}}{1+\beta}}{1 + \frac{\sqrt{1-\beta}}{1+\beta}}$

$$\ell = \frac{1+\beta - \sqrt{1-\beta^2}}{1+\beta+\sqrt{1-\beta^2}}$$

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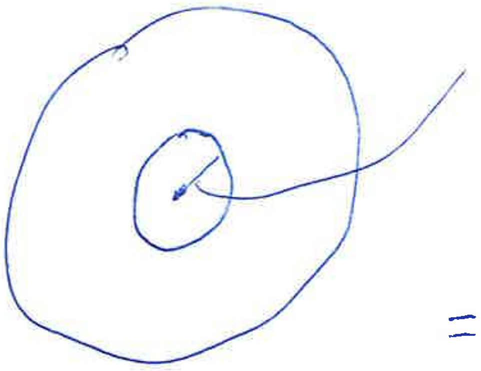

$$\rho = \frac{1+\beta - \sqrt{1-\beta^2}}{1+\beta+\sqrt{1-\beta^2}}.$$

$$= \frac{\sqrt{1+\beta} - \sqrt{1-\beta}}{\sqrt{1+\beta}+\sqrt{1-\beta}}$$

X

$$x = \frac{1+u}{1-u} = \left(\frac{1+w}{1-w}\right)\left(\frac{1+v}{1-v}\right)$$

Then $1+u = x - xu$ so $u = \frac{x-1}{x+1}$

$$= \frac{(1+w)(1+v) - (1-w)(1-v)}{(1+w)(1+v) + (1-w)(1-v)}$$

$$= \frac{2w + 2v}{2 + 2wv} = \frac{w+v}{1+wv}$$

Hence. $u = \frac{w+v}{1+wv}$

goes to multiplicative law. $\tilde{K}_u = \tilde{K}_w \tilde{K}_v$

$$\frac{1+u}{1-u} = \left(\frac{1+w}{1-w}\right)\left(\frac{1+v}{1-v}\right)$$

or to additive law

$$\ln \frac{1+u}{1-u} = \ln \frac{1+w}{1-w} + \ln \frac{1+v}{1-v}. \hookrightarrow \underline{\text{rapidities}}$$

$$\ln K_u = \ln K_w + \ln K_v.$$